

T-Duality in Affine NA Toda Models

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The construction of Non Abelian affine Toda models is discussed in terms of its underlying Lie algebraic structure. It is shown that a subclass of such non conformal two dimensional integrable models naturally leads to the construction of a pair of actions which share the same spectra and are related by canonical transformations.

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1 Introduction

The affine Toda models consists of a class of relativistic two dimensional integrable models admitting soliton solutions with non trivial topological charge (e.g. the abelian affine Toda models). Among such models we encounter certain Non Abelian affine (NA) Toda models admitting electrically charged solitons [1]. In general, the NA Toda models admit solitons with non trivial internal symmetry structure. The formulation and classification of such models with its global symmetry structure is given in terms of the decomposition of an underlying Lie algebraic structure according to a grading operator Q and, in terms of a pair of constant generators ϵ_{\pm} of grade ± 1 . In particular, integrable perturbations of the WZW model characterized by ϵ_{\pm} describe the dynamics of fields parametrizing the zero grade subalgebra \mathcal{G}_0 . The action manifests chiral symmetry associated to the subalgebra $\mathcal{G}_0^0 \subset \mathcal{G}_0$ due to the fact that $Y \in \mathcal{G}_0^0, [Y, \epsilon_{\pm}] = 0$. The existence of such subalgebra allows the implementation of subsidiary constraints within \mathcal{G}_0^0 and the reduction of the model from the group G_0 to the coset G_0/G_0^0 . The structure of the coset G_0/G_0^0 viewed according to axial or vector gauging leads to different parametrizations and different actions, namely axial or vector actions.

We first discuss the general construction of NA Toda models in terms of the gauged WZW model. Next, we discuss the structure of the coset $G_0/G_0^0 = SL(2) \otimes U(1)^{n-1}/U(1)$ and $G_0/G_0^0 = SL(3)/SL(2) \otimes U(1)$ according to axial and vector gaugings and explicitly construct the associated lagrangians. Finally, we show that the axial and vector models are related by canonical transformation (see [2] and refs. therein) preserving the Hamiltonian which also interchanges the topological and electric charges.

2 General Construction of Toda Models

The basic ingredient in constructing Toda models is the decomposition of a Lie algebra \mathcal{G} of finite or infinite dimension in terms of graded subspaces defined according to a grading operator Q ,

$$[Q, \mathcal{G}_l] = l\mathcal{G}_l, \quad \mathcal{G} = \oplus \mathcal{G}_l, \quad [\mathcal{G}_l, \mathcal{G}_k] \subset \mathcal{G}_{l+k}, \quad l, k = 0, \pm 1, \dots \quad (1)$$

In particular, the zero grade subspace \mathcal{G}_0 plays an important role since it is parametrized by the Toda fields. The grading operator Q induces the notion of negative and positive grade subalgebras and henceforth the decomposition of a group element in the Gauss form, $g = NBM$, where $N = \exp(\mathcal{G}_{<})$, $B = \exp(\mathcal{G}_0)$ and $M = \exp(\mathcal{G}_{>})$. The action for the Toda fields is constructed from the gauged Wess-Zumino-Witten (WZW) action,

$$S_{G/H}(g, A, \bar{A}) = S_{WZW}(g) - \frac{k}{2\pi} \int d^2x \text{Tr}(A(\bar{\partial}gg^{-1} - \epsilon_+) + \bar{A}(g^{-1}\partial g - \epsilon_-) + Ag\bar{A}g^{-1}) \quad (2)$$

where $A = A_- \in \mathcal{G}_{<}$, $\bar{A} = \bar{A}_+ \in \mathcal{G}_{>}$, ϵ_{\pm} are constant operators of grade ± 1 . The action (2) is invariant under

$$g' = \alpha_- g \alpha_+, \quad A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial \alpha_-^{-1}, \quad \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \bar{\partial} \alpha_+^{-1} \alpha_+, \quad (3)$$

where $\alpha_- \in \mathcal{G}_{<}$, $\alpha_+ \in \mathcal{G}_{>}$. It therefore follows that $S_{G/H}(g, A, \bar{A}) = S_{G/H}(B, A', \bar{A}')$.

Integrating over the auxiliary fields A, \bar{A} , we find the effective action,

$$S_{eff}(B) = S_{WZW}(B) - k\phi 2\pi \int \text{Tr}(\epsilon_+ B \epsilon_- B^{-1}) d^2x \quad (4)$$

The equations of motion are given by

$$\bar{\partial}(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0, \quad \partial(\bar{\partial}BB^{-1}) - [\epsilon_+, B\epsilon_- B^{-1}] = 0 \quad (5)$$

It is straightforward to derive from the eqns. of motion (5) that chiral currents are associated to the subalgebra $\mathcal{G}_0^0 \subset \mathcal{G}_0$ defined as $\mathcal{G}_0^0 = \{X \in \mathcal{G}_0, [X, \epsilon_{\pm}] = 0\}$, i.e.,

$$J_X = \text{Tr}(XB^{-1}\partial B), \quad \bar{J}_X = \text{Tr}(X\bar{\partial}BB^{-1}), \quad \bar{\partial}J_X = \partial\bar{J}_X = 0 \quad (6)$$

For the cases where $\mathcal{G}_0^0 \neq 0$, we may impose consistently the additional constraints $J_X = \bar{J}_X = 0, X \in \mathcal{G}_0^0$. The construction of the gauged WZW action taking into account the subsidiary constraints (6) reduces the model from the group G_0 to the coset G_0/G_0^0 and its action is given by

$$S_{G_0/G_0^0}(B, A_0, \bar{A}_0) = S_{WZW}(B) - k\phi 2\pi \int \text{Tr}(\epsilon_+ B \epsilon_- B^{-1}) d^2x - k\phi 2\pi \int \text{Tr}(\pm A_0 \bar{\partial}BB^{-1} + \bar{A}_0 B^{-1}\partial B \pm A_0 B \bar{A}_0 B^{-1} + A_0 \bar{A}_0) d^2x \quad (7)$$

where the \pm signs correspond to axial or vector gaugings respectively. The action (7) is invariant under

$$B' = \alpha_0 B \alpha'_0, \quad A'_0 = A_0 - \alpha_0^{-1} \partial \alpha_0, \quad \bar{A}'_0 = \bar{A}_0 - \bar{\partial} \alpha'_0 (\alpha'_0)^{-1} \quad (8)$$

where $\alpha'_0 = \alpha_0(z, \bar{z}) \in \mathcal{G}_0^0$ for axial and $\alpha'_0 = \alpha_0^{-1}(z, \bar{z}) \in \mathcal{G}_0^0$ for vector cases, i.e., $S_{G_0/G_0^0}(B, A_0, \bar{A}_0) = S_{G_0/G_0^0}(\alpha_0 B \alpha'_0 = g_0^f, A'_0, \bar{A}'_0)$

3 The structure of the coset G_0/G_0^0

We now discuss the structure of the coset G_0/G_0^0 constructed according to axial and vector gaugings. We shall be considering first the NA Toda models where $\mathcal{G}_0^0 = U(1)$. The group element of the zero grade subgroup G_0 is parametrized as

$$B = e^{\tilde{\chi} E_{-\alpha_1}} e^{R l_1 \cdot H + \sum_{l=1}^n \varphi_l h_l} e^{\tilde{\psi} E_{\alpha_1}} \quad (9)$$

According to the axial gauging we can write B as an element of the the zero grade subgroup G_0 is parametrized as

$$B = e^{1\phi 2 R l_1 \cdot H} (g_{0,ax}^f) e^{1\phi 2 R l_1 \cdot H}, \quad g_{0,ax}^f = e^{\tilde{\chi} e^{1\phi 2 R} E_{-\alpha_1}} e^{\sum_{l=2}^n \varphi_l h_l} e^{\tilde{\psi} e^{1\phi 2 R} E_{\alpha_1}} \quad (10)$$

The effective action is obtained integrating (7) over A_0, \bar{A}_0 , yielding [1]

$$\mathcal{L}_{eff}^{ax} = 1\phi 2 \sum_{a,b=2}^n \eta_{ab} \partial \varphi_a \bar{\partial} \varphi_b + 1\phi 2 \bar{\partial} \tilde{\psi} \partial \tilde{\chi} \phi \Delta e^{-\varphi^2} - V_{ax}, \quad \Delta = 1 + n + 1\phi 2 n \psi \chi e^{-\varphi^2} \quad (11)$$

where $\psi = \tilde{\psi} e^{1\phi 2 R}$, $\chi = \tilde{\chi} e^{1\phi 2 R}$, and $V_{ax} = \sum_{l=2}^n e^{2\varphi_l - \varphi_{l-1} - \varphi_{l+1}} + e^{\varphi^2 + \varphi_n} (1 + \psi \chi e^{-\varphi^2})$.

The vector gauging can be implemented from the zero grade subgroup G_0 written as

$$B = e^{u l_1 \cdot H} (g_{0,vec}^f) e^{-u l_1 \cdot H}, \quad \text{where} \quad g_{0,vec}^f = e^{\tilde{\chi} e^u E_{-\alpha_1}} e^{\sum_{l=1}^n \phi_l h_l} e^{\tilde{\psi} e^{-u} E_{\alpha_1}} \quad (12)$$

Since u is arbitrary, we may choose $u = 1\phi 2 l n (\tilde{\psi} \phi \tilde{\chi})$ so that

$$g_{0,vec}^f = e^{t E_{-\alpha_1}} e^{\sum_{l=1}^n \phi_l h_l} e^{t E_{\alpha_1}}, \quad t^2 = \tilde{\psi} \tilde{\chi} \quad (13)$$

The effective action for the vector model is [2]

$$\mathcal{L}_{eff}^{vec} = 1\phi 2 \sum_{a,b=1}^n \eta_{ab} \partial \phi_a \bar{\partial} \phi_b + \partial \phi_1 \bar{\partial} \phi_1 \phi t^2 e^{\varphi^2 - 2\phi_1} + \partial \phi_1 \bar{\partial} \ln(t) + \bar{\partial} \phi_1 \partial \ln(t) - V_{vec}.$$

We now discuss the simplest case in which \mathcal{G}_0^0 is nonabelian, i.e. $\mathcal{G} = \hat{SL}(3)$, $Q = d$, the homogeneous gradation and $\epsilon_{\pm} = l_2 \cdot H^{(\pm 1)}$. In this case $\mathcal{G}_0^0 = SL(2) \otimes U(1)$ is generated by $\mathcal{G}_0^0 = \{E_{\pm \alpha_1}, H_1, H_2\}$ and B is written as

$$\begin{aligned} B &= e^{\tilde{\chi}_1 E_{-\alpha_1}} e^{1\phi 2 (l_1 \cdot H R_1 + l_2 \cdot H R_2)} (g_{0,ax}^f) e^{1\phi 2 (l_1 \cdot H R_1 + l_2 \cdot H R_2)} e^{\tilde{\psi}_1 E_{\alpha_1}} \\ g_{0,ax}^f &= e^{\chi_1 E_{-\alpha_1 - \alpha_2} + \chi_2 E_{-\alpha_2}} e^{\psi_1 E_{\alpha_1 + \alpha_2} + \psi_2 E_{\alpha_2}} \end{aligned} \quad (14)$$

where $l_i, i = 1, 2$ are the fundamental weights of $SL(3)$. The effective action is then obtained by integration over the auxiliary matrix fields A_0, \bar{A}_0 yielding

$$\begin{aligned} \mathcal{L}_{eff}^{ax} = & 1\phi\Delta(\bar{\partial}\psi_2\partial\chi_2(1+\psi_1\chi_1+\psi_2\chi_2)+\bar{\partial}\psi_1\partial\chi_1(1+\psi_2\chi_2)) \\ & -1\phi2\Delta(\psi_2\chi_1\bar{\partial}\psi_1\partial\chi_2+\chi_2\psi_1\bar{\partial}\psi_2\partial\chi_1)-V \end{aligned} \quad (15)$$

where $V = 2\phi3 + \psi_1\chi_1 + \psi_2\chi_2$ and $\Delta = (1 + \psi_2\chi_2)^2 + \psi_1\chi_1(1 + 3\phi4\psi_2\chi_2)$. For the vector action, the zero grade group element B in (14) is parametrized as

$$B = e^{\tilde{\chi}_1 E_{-\alpha_1}} e^{1\phi2(l_1 \cdot Hu_1 + l_2 \cdot Hu_2)} (g_{0,vec}^f) e^{-1\phi2(l_1 \cdot Hu_1 + l_2 \cdot Hu_2)} e^{\tilde{\psi}_1 E_{\alpha_1}} \quad (16)$$

where $g_{0,vec}^f = e^{-t_2 E_{-\alpha_2} - t_1 E_{-\alpha_1 - \alpha_2}} e^{\phi_1 h_1 + \phi_2 h_2} e^{t_2 E_{\alpha_2} + t_1 E_{\alpha_1 + \alpha_2}}$. The effective action is then [3]

$$\begin{aligned} \mathcal{L}_{vec} = & 1\phi2 \sum_{i=1}^2 \eta_{ij} \partial\phi_i \bar{\partial}\phi_j + \partial\phi_1 \bar{\partial}\phi_1 \phi t_1^2 e^{-\phi_1 - \phi_2} + \bar{\partial}\phi_1 \partial \ln(t_1) + \partial\phi_1 \bar{\partial} \ln(t_1) \\ & - \partial\phi_1 \bar{\partial}\phi_1 (t_2 \phi t_1)^2 e^{-2\phi_1 + \phi_2} + \bar{\partial}(\phi_2 - \phi_1) \partial(\phi_2 - \phi_1) \phi t_2^2 e^{\phi_1 - 2\phi_2} \\ & + \bar{\partial}(\phi_2 - \phi_1) \partial \ln(t_2) + \partial(\phi_2 - \phi_1) \bar{\partial} \ln(t_2) - V \end{aligned} \quad (17)$$

where $V = 2\phi3 - t_2^2 e^{-\phi_1 + 2\phi_2} - t_1^2 e^{\phi_1 + \phi_2}$ and $\eta_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$.

4 Axial-Vector Duality

In this section we shall prove that the axial and vector models are related by a canonical transformation. Consider the $SL(3)$ vector model

$$\begin{aligned} \mathcal{L}_{vec} = & \partial\phi_1 \bar{\partial}\phi_1 + \partial\phi_2 \bar{\partial}\phi_2 - 1\phi2\partial\phi_2 \bar{\partial}\phi_1 - 1\phi2\partial\phi_1 \bar{\partial}\phi_2 + \partial\phi_1 \bar{\partial}\phi_1 \phi t^2 e^{\varphi_2 - 2\phi_1} \\ & + \partial\phi_1 \bar{\partial} \ln(t) + \bar{\partial}\phi_1 \partial \ln(t) - (e^{\phi_1 - 2\phi_2} + e^{-\phi_1 + 2\phi_2} - t^2 e^{\phi_1 + \phi_2}). \end{aligned} \quad (18)$$

In terms of the new set of more convenient variables $a = (1 - t^2 e^{2\phi_1 - \phi_2})$, $f = \phi_1 - 2\phi_2$, $\theta = \phi_1$ the lagrangian (18) becomes

$$4\mathcal{L}_{vec} = (1 + 3a\phi1 - a)\partial\theta\bar{\partial}\theta - \partial\theta(\bar{\partial}f + 2\bar{\partial}a\phi1 - a) - \bar{\partial}\theta(\partial f + 2\partial a\phi1 - a) + \partial f \bar{\partial}f - 4V_{vec} \quad (19)$$

where $V_{vec} = e^f + ae^{-f}$. The canonical momenta are given by $\Pi_\rho = \delta\mathcal{L}_{vec}/\delta\dot{\rho}$, $\rho = \theta, f, a$. The hamiltonian is then given by

$$\begin{aligned} \mathcal{H}_{vec} = & -(1-a)\Pi_a\Pi_\theta + \Pi_f^2 - (1-a)\Pi_a\Pi_f - a(1-a)\Pi_a^2 \\ & + 1\phi4(1+3a)\phi1 - a\theta'^2 - 1\phi2(f' + 2a'\phi1 - a)\theta' - 1\phi4f'^2 + V_{vec} \end{aligned} \quad (20)$$

Consider now the following modified lagrangian

$$\mathcal{L}_{mod} = \mathcal{L}_{vec} - \tilde{\theta}(\partial\bar{P} - \bar{\partial}P) \quad (21)$$

where we identify $\partial\theta = P$, $\bar{\partial}\theta = \bar{P}$ [4]. Integrating by parts,

$$\begin{aligned} \mathcal{L}_{mod} = & (1 + 3a\phi1 - a)P\bar{P} - 1\phi4P(\bar{\partial}f + 2\bar{\partial}a\phi1 - a + \bar{\partial}\tilde{\theta}) \\ & - 1\phi4\bar{P}(\partial f + 2\partial a\phi1 - a - \partial\tilde{\theta}) + 1\phi4\partial f \bar{\partial}f - V_{vec} \end{aligned} \quad (22)$$

Integrating over the auxiliary fields P and \bar{P} we find the effective action

$$\mathcal{L}_{eff} = 1\phi 4\partial f \bar{\partial} f - 1\phi 4(1-a)\phi 1 + 3a(\bar{\partial} f + 2\bar{\partial} a\phi 1 - a + \bar{\partial} \tilde{\theta})(\partial f + 2\partial a\phi 1 - a - \partial \tilde{\theta}) - V \quad (23)$$

with canonical momenta defined by $\Pi_\rho = \delta \mathcal{L}_{eff} / \delta \dot{\rho}$, $\rho = \tilde{\theta}, f, a$. The hamiltonian becomes

$$\begin{aligned} \mathcal{H}_{mod} = & -1\phi 2(1-a)\Pi_a \tilde{\theta}' + \Pi_f^2 - (1-a)\Pi_a \Pi_f - a(1-a)\Pi_a^2 \\ & + (1+3a)\phi 1 - a\Pi_{\tilde{\theta}}^2 - 1\phi 2(f' + 2a'\phi 1 - a)\Pi_{\tilde{\theta}} - 1\phi 4f'^2 + V_{vec} \end{aligned} \quad (24)$$

The canonical transformation

$$\Pi_\theta = -1\phi 2\tilde{\theta}', \quad \theta' = -2\Pi_{\tilde{\theta}} \quad (25)$$

preserves the Poisson bracket structure and provide the equality of the hamiltonians $\mathcal{H}_{mod} = \mathcal{H}_{vec}$. If we now substitute

$$\tilde{\theta} = 2\ln(\psi\phi\chi), \quad a = 1 + \psi\chi e^{-\varphi^2}, \quad f = -\varphi_2 \quad (26)$$

in the effective lagrangian \mathcal{L}_{eff} (23), we find

$$\mathcal{L}_{eff} = 1\phi 2\partial\varphi_2 \bar{\partial}\varphi_2 + \partial\chi \bar{\partial}\psi\phi\Delta e^{-\varphi^2} - V \quad (27)$$

which is precisely the axial lagrangian (11) for $\mathcal{G}_0 = SL(3)$. It therefore becomes clear that the axial and the vector models are related by the canonical transformation (25) which preserves their hamiltonians.

For the case of $\mathcal{G}_0^0 = SL(2) \otimes U(1)$ we found the canonical transformation responsible for the equality of the Hamiltonians to be [3]

$$\Pi_{\theta_\alpha} = -2\partial_x \tilde{\theta}_\alpha, \quad \Pi_{\tilde{\theta}_\alpha} = -2\partial_x \theta_\alpha, \quad \alpha = 1, 2 \quad (28)$$

where $\theta_\alpha = \ln(\psi_\alpha/\chi_\alpha)$, $\tilde{\theta}_1 = -1\phi 2\phi_1$, $\tilde{\theta}_2 = -1\phi 2(\phi_1 + \phi_2)$

As a last comment of this section we should like to analyse the topological and Noether charges of the axial and vector models. Consider the first example where $\mathcal{G}_0^0 = U(1)$ and with vector and axial lagrangians given by (19) and (23) respectively. The Noether charges associated to the global transformation $\theta \rightarrow \theta + c$ and $\tilde{\theta} \rightarrow \tilde{\theta} + \tilde{c}$ where $c, \tilde{c} = \text{constant}$, are given by

$$\begin{aligned} Q_{vec}^{Noether} &= \int (\delta \mathcal{L}_{vec} \phi \delta \dot{\theta}) \delta \theta dx = \int \Pi_\theta dx, \\ Q_{ax}^{Noether} &= \int (\delta \mathcal{L}_{ax} \phi \delta \dot{\tilde{\theta}}) \delta \tilde{\theta} dx = \int \Pi_{\tilde{\theta}} dx \end{aligned} \quad (29)$$

Since the vector and axial models possess respectively the following topological charges

$$Q_{vec}^{Top} = \int (\partial_x \theta) dx, \quad Q_{ax}^{Top} = \int (\partial_x \tilde{\theta}) dx \quad (30)$$

it is clear that under the canonical transformation (25) their Noether and topological charges become interchanged. The same can be extended to all isometric variables within the models described by lagrangians (15) and (17).

5 Concluding Remarks

We have seen that the crucial ingredient which allows the construction of the axial and vector models is the existence of a non trivial subgroup G_0^0 . We have worked out explicitly examples in which $G_0^0 = U(1)$ and $G_0^0 = SL(2) \otimes U(1)$ involving one and two isometric variables θ_α . The same strategy works equally well for generalized multicharged NA Toda models.

An interesting and intriguing subclass of NA Toda models correspond to the following three affine Kac-Moody algebras, $B_n^{(1)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. Their axial and vector actions were constructed in [2] and shown to be identical. In fact, those affine algebras satisfy the *no torsion condition* proposed in [2] which is fulfilled by Lie algebras possessing B_n -tail like Dynkin diagrams. The very same selfdual models were shown to possess an exact S-matrix coinciding with certain Thirring models coupled to affine abelian Toda models in ref. [5].

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References

- [1] J.F. Gomes, E.P. Gueuvoghlanian, G.M. Sotkov and A.H. Zimerman, *Nucl. Phys.* **B598** (2001) 615, hep-th/0011187; *Nucl. Phys.* **B606** (2001) 441, hep-th/0007169
- [2] J.F. Gomes, E.P. Gueuvoghlanian, G.M. Sotkov and A.H. Zimerman, *Ann. of Phys.* **289** (2001) 232, hep-th/0007116
- [3] J.F. Gomes, G.M. Sotkov and A.H. Zimerman, *J. Physics* **A37** (2004) 4629
- [4] T. Busher, *Phys. Lett.* **159B** (1985) 127, *Phys. Lett.* **194B** (1987) 59, *Phys. Lett.* **201B** (1988) 466
- [5] V.A. Fateev, *Nucl. Phys.* **B479** (1996) 594